ANALYTICAL SOLUTIONS OF THE TIME-FRACTIONAL NON-LINEAR SCHRODINGER EQUATION WITH ZERO AND NON ZERO TRAPPING POTENTIAL THROUGH THE SUMUDU DECOMPOSITION METHOD

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ABSTRACT

Sumudu decomposition method is used to construct the approximate analytical solutions of time-fractional nonlinear Schrodinger equations with zero and nonzero trapping potential. The Sumudu decomposition method is a combined form of the Sumudu transform and the Adomian decomposition method. The fractional derivatives are defined in the Caputo sense. The exact solutions of some nonlinear Schrodinger equations are given as a special case of our approximate analytical solutions. The computations show that the described method is easy to apply, and it needs smaller size of computation as compared to the other existing methods. Further, the solutions are derived in a convergent series form which shows the effectiveness of the method for solving a wide variety of nonlinear fractional differential equations.

Keywords: Fractional derivative; Time-fractional Schrodinger equation; Adomian decomposition method; Sumudu transforms.

INTRODUCTION

The time-fractional nonlinear Schrodinger equations has the following form:

$$iD_t^\alpha u(x,t) = -\frac{1}{2}\Delta^2 u + V_d(x)u + \beta_d |u|^2 u, \quad 0 < \alpha \leq 1,$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d, t > 0, \quad (1)$$

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where $V_d(x)$ is the trapping potential and β_d is a real constant. The physical model (1) and its generalized forms arise in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics. Finding accurate and efficient methods for solving fractional nonlinear Schrodinger equations has been an active research undertaking. Exact solutions of most of the fractional nonlinear Schrodinger equations cannot be found easily. Therefore, analytical and numerical methods have been used in the literature. Some of the analytical methods for solving nonlinear problems include the Adomian decomposition method (ADM) (Khuri, 1998, Rida *et al.*, 2008), Variational iteration method (VIM) (Wazwaz, 2008), Homotopy-perturbation method (HPM) (Sadighi and Ganji, 2008), Differential transformation method (DTM) (Ravi Kanth and Aruna, 2009) and Homotopy analysis method (HAM) (Liao, 2003).

Adomian decomposition method is a semi-analytical method introduced by Adomian (1980), which provides an effective technique for finding explicit of a wider and general class of differential systems representing real physical problems (Ali and Al-Saif, 2008). This method and its modifications efficiently work for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations (Mahmoud and Borai, 1980). Moreover, no linearization or perturbation is required in this method.

In this study, we use the Sumudu decomposition method (SDM) to construct the approximate analytical solutions of time-fractional nonlinear Schrodinger equations with zero and nonzero trapping potential. The SDM is a combined form of the Sumudu transform and the Adomian decomposition method. The Sumudu transform is a new integral transform similar to the Laplace transform, introduced by Watugala (1993) to solve differential equations.

Our manuscript is organized as follows: In section II, we present the brief summary of Sumudu decomposition method. Results and discussion is given in Section III. In the following two subsections, the analytical solution of the time-fractional nonlinear Schrodinger equations with zero and nonzero trapping potential are provided. Finally, the concluding remark to the proposed results is given in Section IV.

SUMUDU DECOMPOSITION METHOD

In this section, we implement the Adomian decomposition method with Sumudu transformation to time-fractional nonlinear Schrodinger equations with zero and nonzero trapping potential given by Eq. (1). The Adomian decomposition method requires that the nonlinear fractional differential Eq. (1) be expressed in terms of operator from as

$$D_t^\alpha u(x, t) = -i \left(-\frac{1}{2} L_x u + V(x)u + \beta_d g(u) \right), \quad (2)$$

where $L_x = \frac{\partial^2}{\partial x^2}$ is a linear operator, D_t^α is the fractional differential operator and nonlinear function $g(u) = |u|^2 u$. The methodology consists of applying Sumudu transform first on both sides of Eq. (2), obtaining

$$S[u(x, t)] = u(x, 0) - i u^\alpha S \left[-\frac{1}{2} L_x u + V(x)u + \beta_d g(u) \right]. \quad (3)$$

Sumudu inverse transform of Eq. (3), gives

$$u(x, t) = u(x, 0) - i S^{-1} \left[u^\alpha S \left[-\frac{1}{2} L_x u + V(x)u + \beta_d g(u) \right] \right] \quad (4)$$

The Adomian decomposition method suggests the solution $u(x, t)$ be decomposed into the infinite series of components

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad (5)$$

and the nonlinear function in Eq.(2) is decomposed as follows:

$$g(u) = \sum_{n=0}^{\infty} A_n, \quad (6)$$

where A_n are so-called the Adomian polynomials. Substitution the decomposition series (5) and (6) into both sides of (4) gives,

$$\sum_{n=0}^{\infty} u_n(x, t) = u(x, 0) - i S^{-1} \left[u^\alpha S \left[-\frac{1}{2} L_x \left(\sum_{n=0}^{\infty} u_n(x, t) \right) + V(x) \left(\sum_{n=0}^{\infty} u_n(x, t) \right) + \beta_d \sum_{n=0}^{\infty} A_n \right] \right] \quad (7)$$

From this equation, the iterates are determined by the following recursive way

$$\begin{aligned}
u_0(x, t) &= u(x, 0), \\
u_1(x, t) &= -i S^{-1} \left[u^\alpha S \left[-\frac{1}{2} L_x u_0 + V(x) u_0 + \beta_d A_0 \right] \right], \\
u_2(x, t) &= -i S^{-1} \left[u^\alpha S \left[-\frac{1}{2} L_x u_1 + V(x) u_1 + \beta_d A_1 \right] \right], \\
&\dots\dots\dots, \\
u_{n+1}(x) &= -i S^{-1} \left[u^\alpha S \left[-\frac{1}{2} L_x u_n + V(x) u_n + \beta_d A_n \right] \right].
\end{aligned} \tag{8}$$

The Adomian polynomial A_n can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian (1994) or alternatively by Wazwaz (2000).

The general form of formula for A_n Adomian polynomials is

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[g \left(\sum_{k=0}^n \lambda^k u_k \right) \right]_{\lambda=0} \tag{9}$$

Setting $|u|^2 = u \bar{u}$ the polynomials A_n that represent the nonlinear term $g(u)$ are given by

$$\begin{aligned}
A_0 &= |u_0|^2 u_0, \\
A_1 &= 2|u_0|^2 u_1 + u_0^2 \bar{u}_1, \\
A_2 &= 2|u_0|^2 u_2 + \bar{u}_0 u_1^2 + 2|u_1|^2 u_0 + u_0^2 \bar{u}_2, \\
&\dots\dots\dots
\end{aligned} \tag{10}$$

and so on.

Finally, by substituting the values of $u_1, u_2, u_3, \dots$ from Eq. (8), we get the solution of the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).$$

In this approach, the solution is found in the form of a convergent series. Convergence conditions of this series are examined by several authors, mainly by Cherruault (1993).

RESULTS AND DISCUSSION

In the following two subsections, we construct the approximate analytical solutions to the time-fractional nonlinear Schrodinger equations with zero and nonzero trapping potential by means of Sumudu decomposition method. In order to illustrate the efficiency of this method, we compare our solutions with the exact solutions for some specific choice of parameters.

A. Time-fractional nonlinear Schrodinger equations with zero trapping potential

Consider the following time-fractional nonlinear Schrodinger equations with zero trapping potential (i.e. $V_d(x) = 0$)

$$iD_t^\alpha u(x,t) = -\frac{1}{2}\Delta^2 u + \beta_d |u|^2 u, \quad t > 0, \quad 0 < \alpha \leq 1,$$

$$u(x,0) = e^{ix}, \quad x \in \mathbb{R}^d, \quad (11)$$

by using Eq. (8) and Eq. (10), we terms of the decomposition series could be calculated (5) as:

$$u_0(x,t) = e^{ix},$$

$$u_1(x,t) = \frac{ie^{ix} t^\alpha c_1}{2\Gamma[1+\alpha]},$$

$$u_2(x,t) = \frac{-e^{ix} t^{2\alpha} (1+2\beta_d) c_2}{4\Gamma[1+2\alpha]}, \quad (12)$$

$$u_3(x,t) = \frac{ie^{ix} t^{3\alpha} (1+2\beta_d)^2 c_3}{8\Gamma[1+3\alpha]},$$

$$u_4(x,t) = \frac{ie^{ix} t^{4\alpha} (1+2\beta_d)^3 c_4}{16\Gamma[1+4\alpha]},$$

.....and so on.

Substituting $u_0(x, t)$, $u_1(x, t)$, $u_2(x, t)$. . . into (5) yields the solution

$$u(x, t) = e^{ix} + \sum_{n=1}^{\infty} \frac{(-i)^n e^{ix} t^{n\alpha} (1+2\beta_d)^{n-1} c_n}{2^n \Gamma[1+n\alpha]} \quad (13)$$

where the constraints read as:

$$c_1 = c_2 = (1+2\beta),$$

$$c_3 = (c_1 + 4\beta) - \frac{2\beta \Gamma[1+2\alpha]}{\Gamma[1+\alpha]^2}, \quad (14)$$

$$c_4 = c_3 - \frac{2\beta_d \Gamma[1+3\alpha]}{\Gamma[1+\alpha]^3} + \frac{4\beta_d \Gamma[1+3\alpha]}{\Gamma[1+\alpha]\Gamma[1+2\alpha]}.$$

.....and so on.

Now, we analyse in details the above solution in two different cases.

Case 1. $\alpha = 1$ and $\beta_d = -1$.

In this case, Eq. (12) becomes,

$$u(x, t) = e^{ix} + \sum_{n=1}^{\infty} \frac{(it)^n e^{ix}}{2^n n!} = e^{i(x+1/2t)} \quad (15)$$

which is an exact solution to the nonlinear cubic Schrodinger equation (Khuri, 1998).

Case 2. $\alpha = 1/2$ and $\beta_d = -1$.

In this case, we do not have the exact analytic solution of the fractional Schordinger equation. We list the 5-term approximation as follows,

$$u(x, t) = e^{ix} + \frac{ie^{ix} t^{1/2}}{\pi^{1/2}} - \frac{ie^{ix} t}{4} + \frac{ie^{ix} (8-5\pi)t^{3/2}}{6\pi^{3/2}} - \frac{ie^{ix} (20-11\pi)t^2}{32\pi}. \quad (16)$$

We plotted the graphs for both cases. Figure 1, 2 lead to the understanding that the behavior of the approximate solutions are the same behavior with the exact solution. Consequently, the approximate solutions are rapidly convergent as the exact solutions. In order to illustrate the efficiency of the SDM, we compared the approximate numerical solution with the exact solutions in Table 1. The numerical results show that SDM yields a good approximate solution, even if use lowest order approximate solution. The accuracy can be improved by using higher order approximate solutions.

Figure 1: The real part of the exact solution (15) shown in the figure (a) in comparison with the real part of the 5th order approximate solution (13) shown in (b) when $\alpha = 1$, $\beta_d = -1$, $-10 < x < 10$ and $0 < t < 1$.

Figure 2: The imaginary part of the exact solution (15) shown in the figure (a) in comparison with the imaginary part of the 5th order approximate solution (13) shown in (b) when $\alpha = 1$, $\beta_d = -1$, $-10 < x < 10$ and $0 < t < 1$.

Table 1: Comparison of the real part of exact solution with 4th order approximate solution for $t=0.7$.

X	U _{SDM}	U _{Exact}	Abs. Error $ U_{SDM}-U_{Exact} $
1	0.21904479	0.21900669	0.0000381
2	-0.70267445	-0.70271308	0.00003862
3	-0.97835804	-0.97836168	0.00003637
4	-0.35454376	-0.35450907	0.00003469
5	0.59523642	0.59527755	0.00004113

B. Time-fractional nonlinear Schrodinger equations with trapping potential

Consider the following time-fractional nonlinear Schrodinger equations with trapping potential (i.e. $V_d(x) = \cos^2 x$ and $\beta_d = 1$)

$$iD_t^\alpha u(x,t) = -\frac{1}{2}\Delta^2 u + V_d(x)u + |u|^2 u, \quad t > 0, \quad 0 < \alpha \leq 1,$$

$$u(x,0) = \sin x, \quad x \in \mathbb{R}^d, \quad (17)$$

by using Eq. (8) and Eq. (10), the terms of the decomposition series (5) could be calculated as:

$$u_0(x,t) = \sin x,$$

$$u_1(x,t) = -\frac{3it^\alpha \sin x f_1(x)}{2\Gamma[1+\alpha]},$$

$$u_2(x,t) = -\frac{27t^{2\alpha} \sin x f_2(x)}{4\Gamma[1+2\alpha]}, \quad (18)$$

$$u_3(x,t) = -\frac{81it^{3\alpha} \sin x f_3(x)}{8\Gamma[1+3\alpha]},$$

$$u_4(x,t) = \frac{9t^{4\alpha} \sin x f_4(x)}{16\Gamma[1+4\alpha]},$$

.....and so on.

Substituting $u_0(x,t)$, $u_1(x,t)$, $u_2(x,t)$. . . into (5) yields the solution

$$u(x,t) = \sin x + \sum_{n=1}^{\infty} \frac{(-3i)^n t^{n\alpha} \sin x f_n(x)}{2^n \Gamma[1+n\alpha]} \quad (19)$$

where the constraints read as:

$$f_1(x) = f_2(x) = 1,$$

$$f_3(x) = \frac{(5-2\cos 2x)}{3} - \frac{2\Gamma[1+2\alpha]\sin^2 x}{3\Gamma[1+\alpha]^2}, \quad (20)$$

$$f_4(x) = \frac{3-25\cos 2x + \cos 4x}{9} + \frac{2(2\Gamma[1+\alpha]^2 - \Gamma[1+2\alpha])\Gamma[1+3\alpha]\sin^2 x}{3\Gamma[1+\alpha]^3 \Gamma[1+2\alpha]} \\ + \frac{\Gamma[1+2\alpha](3+9\cos 2x - 4\sin^4 x)}{9\Gamma[1+\alpha]^2}$$

.....and so on.

Similarly as the zero trapping potential, we analyse in details of the above solution for two different cases.

Case 1. $\alpha = 1$, the approximate solution (19) reduces to

$$u(x,t) = \sin x + \sum_{n=1}^{\infty} \frac{(-3it)^n \sin x}{2^n n!} = \sin x e^{-3/2it}, \quad (21)$$

which is exact solution obtained by differential transform method (Ravi Kanth and Aruna, 2009).

Case 2. $\alpha = 1/2$, we do not have the exact analytic solution. We list the 4-term approximation as follows,

$$u(x,t) = \sin x - \frac{3it^{1/2} \sin x}{\pi^{1/2}} - \frac{9}{4} t \sin x - \frac{3it^{3/2} (4-5\pi + 2(-2+\pi)\cos 2x) \sin x}{32\pi}. \quad (22)$$

Similarly as for zero trapping potential, the approximate solutions are having the same behavior with the exact solution and convergence as the exact solution. This is

illustrated in Figures 3 and 4. In Table.2, we compare the exact solution with the 4th order approximate solutions by SDM of Eq. (20). The numerical results show that the SDM provides more accurate approximate solutions.

Figure 3: The real part of the exact solution (20) shown in the figure (a) in comparison with the real part of the 4th order approximate solution (19) shown in (b) when $\alpha = 1$, $-10 < x < 10$, and $0 < t < 1$.

Figure 4: The imaginary part of the exact solution (20) shown in the figure (a) in comparison with the imaginary part of the 4th order approximate solution (19) shown in (b) when $\alpha = 1$, $-10 < x < 10$, and $0 < t < 1$.

Table 2: Comparison of the real part of exact solution with 4th order approximate solution for $t=0.3$.

x	U_{SDM}	U_{Exact}	Abs. Error $ U_{SDM}-U_{Exact} $
1	0.75742997	0.75770011	0.00027014
2	0.81860531	0.81877423	0.00016892
3	0.12691518	0.12707110	0.00015592
4	-0.6811010	-0.68146061	0.00035961
5	-0.86338371	-0.86346058	0.00007687

CONCLUSION

In the present paper, the Sumudu decomposition method is applied successfully for finding the approximate analytical solutions of the time-fractional nonlinear Schrodinger equations with zero and nonzero trapping potential that are considered in the Caputo sense. For some special choice of parameters, our approximate solutions are exactly same as the exact solutions of some nonlinear Schrodinger equations. From the Figs. 1–4, we conclude that the behavior of the approximate solutions is the same behavior of the exact solution at some different values of α and β . Consequently, we realize that the approximate solution is rapidly convergent series as the exact solutions. Further, the computations show that the described method is easy to apply and it needs a computational code smaller than that for the existing methods. Therefore, it is obvious that the SDM is very powerful, efficient and accurate for solving the nonlinear fractional differential equations. The computations associated with example in this paper were carried out using Mathematica.

REFERENCES

- Adomian G. (1994). Solving frontier problems of physics: the decomposition method, *Boston: Kluwer Academic Publishers*.
- Ali A.H., Al-Saif A.S.J. (2008). Adomian decomposition method for solving some models of nonlinear Partial Differential Equations, *Basrah Journal of Scienec (A)* **26**, 1-11.
- Belgacem F.B.M., Karaballi A.A. (2006). Sumudu transform fundamental properties investigations and applications. *International Journal of Stochastic Analysis*, 1–23.<http://doi.org/10.1155/JAMSA/2006/91083>
- Caputo M., Mainardi F. (1971). Linear models of dissipation in anelastic solids. *Rivista Del Nuovo Cimento* **1**, 161-198.<http://doi.org/10.1007/BF02820620>
- Cherruault Y, Adomian G. (1993). Decomposition methods: a new proof of convergence, *Mathematical and Computer Modelling*, **18(12)**, 103-106. [https://doi.org/10.1016/0895-7177\(93\)90233](https://doi.org/10.1016/0895-7177(93)90233)
- Khuri S.A (1998). A new approach to the cubic Schrodinger equation: An application of the decomposition technique, *Applied Mathematics and Computation* **97**, 251-254. [http://doi.org/10.1016/S0096-3003\(97\)10147-3](http://doi.org/10.1016/S0096-3003(97)10147-3)
- Liao S. J. (2003). Introduction to the Homotopy Analysis Method Beyond Perturbation, *Chapman and Hall/CRC Press, Boca Raton*.
- Mahmoud M., El-Borai (1980). On the initial value problem for partial differential equations with operator coefficients, *International Journal of Mathematics and Mathematical Sciences*, **3**, 103-111. <http://doi.org/10.1155/S0161171280000075>
- Miller K.S., Ross B. (1993). An Introduction to the Fractional Calculus and Fractional Differential Equations, *New York, Wiley*.
- Ravi Kanth A.S.V., Aruna K. (2009). Two-dimensional differential transform method for solving linear and non-linear Schrodinger equations, *Chaos, Solitons and Fractals*, **41(5)**, 2277-2281. <http://doi.org/10.1016/j.chaos.2008.08.037>
- Rida S.Z., El-Sherbiny H.M., Arafa A.A.M. (2008). On the solution of the fractional nonlinear Schrodinger equation, *Physics Letters A* **372**, 553-558. <https://doi.org/10.1016/j.physleta.2007.06.071>
- Sadighi A., Ganji D.D. (2008). Analytic treatment of linear and nonlinear Schrodinger equations: a study with homotopy-perturbation and Adomian decomposition

methods, *Physics Letters A*, **372** (4), 465-469.
<https://doi.org/10.1016/j.physleta.2007.07.065>

Samko S.G., Kilbas A. A., Marichev O. I. (1993). Fractional Integrals and Derivatives –
Theory and Applications, Gordon and Breach.

Watugala G.K. (1993). Sumudu transform: a new integral transform to solve differential
equations and control engineering problems, *International Journal of
Mathematical Education in Science and Technology*, **24**(1), 35–43.
<https://doi.org/10.1080/0020739930240105>

Wazwaz A.M. (2000). A new algorithm for calculating Adomian polynomials for
nonlinear operators, *Applied Mathematics and Computation*. **111**, 53-69.
[http://doi.org/10.1016/S0096-3003\(99\)00063-6](http://doi.org/10.1016/S0096-3003(99)00063-6)

Wazwaz A.M. (2008). A study on linear and nonlinear Schrodinger equations by the
variational iteration method, *Chaos, Solitons and Fractals*, **37**(4), 1136-1142.
<http://doi.org/10.1016/j.chaos.2006>