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A NEW INTERPRETATION OF PRIMITIVE PYTHAGOREAN TRIPLES AND A CONJECTURE RELATED TO FERMAT'S LAST THEOREM

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ABSTRACT

In this study primitive Pythagorean triples have been carefully examined and found that all of them satisfy a simple rule related to mean value theorem. It is pointed out that integral triples satisfying the equation on Fermat's Last Theorem should satisfy a special rule related to mean value theorem. A conjecture is proposed which may lead to find a simple proof of Fermat's Last Theorem.

Keywords: Pythagorean triples, Fermat's Last Theorem

INTRODUCTION

The equation

$$z^n = y^n + x^n \ , \quad (x, y) = 1 \qquad \textbf{(A)}$$

with primes $(n \neq 2)$ has been highly attracted by mathematicians for about three and half centuries due to Fermat's claim that there is no integral triple (x, y, z) satisfying the equation (A). The proof of the general case is extremely difficult. However, when

$n = 2$, (which is a prime) the equation has solutions and some solutions were known possibly even before $582\,B.C.$ These solutions are called primitive Pythagorean triples and the corresponding form of equation (A) is called Pythagoras equation. It is well known that all primitive Pythagorean triples can be generated (Bolker & Benjamin, 1969) by the generators

$z = a^2 + b^2$, $y = a^2 - b^2$, $x = 2ab$, with integral a, b of opposite parity (even or odd) satisfying $a > b > 0$.

Fermat used these primitive Pythagorean triples to prove his conjecture for $n = 4$. Fermat's conjecture has been proved and the equation (A) has no non-trivial integral solutions (For a proof see Ribenboin, 1979 p 37-38). Therefore we believe that a closer look at primitive Pythagorean triples needed than ever before, along a new direction in order to understand the major difficulty in giving a simple proof for Fermat's last theorem. With this in mind , we have examined the whole set of primitive Pythagorean triples (x, y, z) satisfying $0 < x < y < z$ and we have found that odd pair of (z, x) or (z, y) satisfies $z - x = 2^{2\beta-1}$ or $z - x = 2^{2\beta-1} l^2$ where $l > 1$ and $(2, l) = 1$ and $\dfrac{z + x}{2}$ is a perfect square, which are proved in § 2. The Relation of $(z - x)$ and $(z - y)$ to the mean value theorem is also discussed in this section. It is well known that if we assume that (A) has a non-zero solution (x, y, z), one of x, y, z is divisible by *n*. Then if we assume that *y* is divisible by n, it can be proved that $z - x = n^{\beta n-1}\alpha^n$ where $(z, x) = 1$, $(n, \alpha) = 1$, and $\alpha \geq 1$.

Then the quantity $z^n - x^n$ is related to the mean value theorem by $z^n - x^n = n(z - x)\xi^{n-1}$,

and in the case of Pythagorean triples,

$z^2 - x^2 = 2(z - x)\left(\dfrac{z + x}{2}\right)$ and $\xi = \dfrac{z + x}{2}$ is an integer , and it will be shown in §.2 that ξ is a perfect square. We have examined $z^3 - x^3$ when $z - x = 3^{3\beta-1}\alpha^3$ numerically in a sufficiently large set of integers for fixed α and ξ turned out to be irrational, and our conjecture related to $z^n - x^n$ is discussed in §.2 in detail. §.3 is devoted to a brief discussion focusing on Fermat's last theorem.

§2. A New look at Primitive Pythagorean triples

Primitive Pythagorean triples (x, y, z) are the positive integral solutions of Pythagoras' equation

$$z^2 = y^2 + x^2 \quad , \quad \text{where } (x, y, z) = 1 \qquad \textbf{(1)}$$

If y and x are odd $y^2 + x^2$ is divisible by 2 but not by 2^2**,** and therefore z is never even. Since one of x, y, z is even, it follows that x or y is even. In the following we consider the case y is even.

First, we shall show that $z - x = 2^{2\beta-1}$ or $z - x = 2^{2\beta-1}\alpha^2$ where α is a factor of y and $(2, \alpha) = 1$. It follows from (1) that $y^2 = (z - x)(z + x)$ and z , x are odd, $(z - x)$ is even.

If $(z - x) = 2^t$. then

$$z^2 - x^2 = (x + 2^t)^2 - x^2 = 2^{t+1}(x + 2^{t-1}) = y^2 \quad .. \qquad \textbf{(2)}$$

where $t \geq 1$.

From , $\quad x + 2^{t-1} \quad$ is odd (as x is odd) $\quad . \qquad$ **(3)**

it follows that $\quad t = 2\beta - 1$ for $\beta \in \mathbb{Z}^+$

If $z - x = 2^{2\beta-1}q$, for $q \in \mathbb{Z}^+$ $(q,2) = 1$, we get

$$z^2 - x^2 = 2^{2\beta}q[x + 2^{2\beta-2}q] \quad . \qquad \textbf{(4)}$$

If $(x, q) \neq 1$, then $(z, q) \neq 1$ and it follows that $(z, x) \neq 1$. Hence $(x, q) = 1$.

Then q and $x + 2^{2\beta-2}q$ should be perfect squares. Let $q = \alpha^2$

Then $\quad z - x = 2^{2\beta-1}\alpha^2$ for $\alpha \in \mathbb{Z}^+$ and $(\alpha, 2) = 1 \quad . \qquad$ **(5)**

Now $$z^2 - x^2 = 2 \cdot 2^{2\beta-1} \alpha^2 \frac{(z+x)}{2} \qquad \ldots \qquad \textbf{(6)}$$

i.e. $y = 2^\beta \alpha . \sqrt{\dfrac{z+x}{2}}$ and hence α is a factor of y.

And since $2 \cdot 2^{2\beta-1} \alpha^2$ is a perfect square $\dfrac{(z+x)}{2}$ also should be a perfect square.

For example,

$$13^2 = 12^2 + 5^2 \quad , \; z - x = 2^3 \; , \quad \frac{z+x}{2} = 3^2$$

$$113^2 = 112^2 + 15^2 \quad , \; z - x = 2 \times 7^2 \; , \quad \frac{z+x}{2} = 8^2$$

§ 3. Relation between Primitive Pythagorean Triples and the Mean Value Theorem

The mean value theorem related to our discussion can be written as

$$\frac{f(z) - f(x)}{z - x} = f'(\xi) \qquad . \qquad \textbf{(7)}$$

where $f(z)$ is a differentiable function of real variable z, x is a constant and $x < \xi < z$.

If $f(z) = z^2$ and x is a positive integer, (7) takes the form

$$\frac{z^2 - x^2}{z - x} = 2\xi \qquad . \qquad \textbf{(8)}$$

And this is true for all z with ξ satisfying $x < \xi < z$.

Therefore if z is a positive integer,

$$\begin{aligned} z^2 - x^2 &= 2(z-x)\xi = 2(z-x)\left(x + \frac{1}{2}(z-x) \right) \\ &= 2(z-x)(x + \theta h) \end{aligned}$$

with $\theta = \frac{1}{2}$ *and* $h = z - x$, this is well known as one form of the mean value theorem relevant to our discussion .

All primitive Pythagorean triples (x, y, z) with y even satisfies

$$y^2 = z^2 - x^2 = 2(z - x)(x + \theta h) \tag{9}$$

where $\theta = \frac{1}{2}$ *and* $h = z - x$.

CONJECTURE

If $f(z) = z^n$ and x is a positive integer, relation to mean value theorem is given by

$$z^n - x^n = n(z - x)\xi^{n-1} \tag{10}$$

where $x < \xi < z$,as in the case of primitive Pythagorean triples. Now if we assume that (A) is true for some odd prime n, then it is well known that [A] has a solution of that xyz is divisible by n. In a numerical calculation of ξ for $n = 3$, ξ turned out to be irrational. We, motivated by the above, conjecture that ξ in (10) is irrational for any odd prime n. We have already proved this result in the case of $n = 3$ rigorously and it will be published elsewhere.

DISCUSSION

It is highly expected that a simple proof of Fermat's last theorem is plausible. All the solutions of the equation

$$z^n = y^n + x^n,\ (x, y) = 1$$

for any odd prime n, satisfying the additional condition that $2n + 1$ is a prime, are such that xyz is divisible by n. Simple proofs however are not available even in the case of $n = 3$.

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