J Sci.Univ.Kelaniya **4** (2008): 15-20

EXACT FORMULA FOR THE SUM OF THE SQUARES OF THE BESSEL FUNCTION AND THE NEUMANN FUNCTION OF THE SAME ORDER OF HALF-ODD INTEGER

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ABSTRACT

Sum of the squares of the Bessel function and the Neumann function of the same order of half-odd integer has been found to be very useful in addressing a puzzle in nuclear physics. One approximate formula available in the literature is valid for the complex argument whose real part is greater than zero, and the absolute value of error term is undefined for half-odd integers. Another approximate formula which is valid for all complex arguments has been obtained using sophisticated mathematical method called Barnes' method. However, the error in the formula is very difficult to calculate. We have obtained exact formula for the sum of the squares of Bessel and Neumann functions of the same order of half-odd integers which is valid for all complex arguments, and its proof is also given.

Keywords: Bessel functions, Neumann functions, Anomalous absorption, Partial waves

INTRODUCTION

In case of elastic scattering of neutrons on composite nuclei, it has been found (Kawai & Iseri, 1985) that the S-matrix element becomes zero for a special combination of energy (E), orbital angular momentum (l), total angular momentum (j) and composite target nuclei (A). This phenomenon is called anomalous absorption of neutron partial waves by the nuclear optical potential. This phenomenon occurs is case of elastic proton scattering on composite target nuclei (Iseri & Kawai, 1986) and it has been found (Piyadasa, 1985) that this phenomenon is universal for light ion elastic scattering on composite nuclei. Anomalous absorption of partial waves of light ion scattering shows striking systematic in varies parameter planes. For example, angular momenta corresponding to partial waves absorbed lie on straight lines in the parameter plane $\left(\frac{l}{k},\, A^{\frac{1}{3}}\right)$ in case of neutron elastic scattering. The partial wave U_{lj} of angular momentum l satisfies the relation,

$$\left|U_{lj}(k,r)\right|^2 = j_l{}^2(kr) + \eta_l{}^2(kr)$$

for a zero of S-matrix element, where k is the incident wave number and *r* is the radial distance between the projectile and the target nucleus. $j_l(kr)$ and $\eta_l(kr)$ stand for the spherical Bessel function and the spherical Neumann function respectively.

In finding the origin of systematic of the anomalous absorption of neutron partial waves by the nuclear optical potential, we have found that exact formula for the sum

$$j_l^2(kr) + \eta_l^2(kr) = \left(\frac{\pi}{2kr}\right)\left[J^2_{l+\frac{1}{2}}(kr) + N^2_{l+\frac{1}{2}}(kr)\right]$$

is very important, where $J_{l+\frac{1}{2}}$ and $N_{l+\frac{1}{2}}$ stand for Bessel function and Neumann function respectively. Only approximate formulae are available in the literature for $J^2_{l+\frac{1}{2}}(kr) + N^2_{l+\frac{1}{2}}(kr)$.

One approximate formula for $J^2_{l+\frac{1}{2}}(Z) + N^2_{l+\frac{1}{2}}(Z)$ has been given (Watson, 1944) which is valid for $\mathrm{Re}(Z) > 0$. However, its validity for all complexes *Z* is conjectured (Grad Shteyn & Ruzhik, 1980). Another one has been derived (Watson, 1944) using the sophisticated mathematical method called Barnes' method which is valid for all complex *Z*, but the error term is supposed to be very difficult to estimate.

We have obtain exact formula for the sum of the square of the Bessel function and the Neumann function of the same order of half odd integer which is valid for all complex Z, and its proof is given in the next section.

MATERIALS AND METHODS

In this section, the proof of the formula is given using the method of mathematical induction.

The following five formulae have been proved for all integral *n* and complex *z* (Watson, 1944).

$$j_{n+\frac{1}{2}}(z)+iN_{n+\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\exp(i[z-(n+1)\pi]\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!(2iz)^k} \quad \textbf{(1)}$$

$$N_{n-\frac{1}{2}}(z)+N_{n+\frac{3}{2}}(z)=\frac{(2n+1)}{z}N_{n+\frac{1}{2}}(z) \quad \textbf{(2)}$$

$$N_{n-\frac{1}{2}}(z)-N_{n+\frac{3}{2}}(z)=2N'_{n+\frac{1}{2}}(z) \quad \textbf{(3)}$$

$$J_{n-\frac{1}{2}}(z)+J_{n+\frac{3}{2}}(z)=\frac{(2n+1)}{z}J_{n+\frac{1}{2}}(z) \quad \textbf{(4)}$$

$$J_{n-\frac{1}{2}}(z)-J_{n+\frac{3}{2}}(z)=2J'_{n+\frac{1}{2}}(z) \quad \textbf{(5)}$$

Theorem: For all $z\in C$ and $n\in Z^+$,

$$J^2_{n+\frac{1}{2}}(z)+N^2_{n+\frac{1}{2}}(z)=\frac{2}{\pi z}\sum_{k=0}^{n}\frac{(2k-1)!!(n+k)!}{2^k z^{2k}\, k!(n-k)!} \quad \textbf{(6)}$$

where $(2k-1)!!=1$ when $k=0$.

Proof: From **(6)**, when $n = 0$, $J^2_{\frac{1}{2}}(z)+N^2_{\frac{1}{2}}(z)=\frac{2}{\pi z}$

and when $n = 1$, $J^2_{\frac{3}{2}}(z)+N^2_{\frac{3}{2}}(z)=\frac{2}{\pi z}\left(1+\frac{1}{z^2}\right)$.

From **(1)**, for $n = 0$,

$$J_{\frac{1}{2}}(z)+iN_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\exp(i(z-\pi))=-\sqrt{\frac{2}{\pi z}}\exp(iz) \quad \textbf{(7)}$$

and putting $-i$ in place of i, we get

$$J_{\frac{1}{2}}(z)-iN_{\frac{1}{2}}(z)=-\sqrt{\frac{2}{\pi z}}\exp(-iz) \quad \textbf{(8)}$$

Hence, multiplying **(7)** and **(8),** we get, $J^2_{\frac{3}{2}}(z)+N^2_{\frac{3}{2}}(z)=\frac{2}{\pi z}$**.**

Again from **(1),** we get, $J_{\frac{3}{2}}(z)+iN_{\frac{3}{2}}(z)=\sqrt{\frac{2}{\pi z}}\exp(iz)\left(1+\frac{1}{iz}\right)$

and putting $-i$ in place of i, we get,

$J_{\frac{3}{2}}(z)-iN_{\frac{3}{2}}=\sqrt{\frac{2}{\pi z}}\exp(iz)\left(1-\frac{1}{iz}\right)$ and multiplying these two gives

$$J^2_{\frac{3}{2}}(z)+N^2_{\frac{3}{2}}(Z)=\frac{2}{\pi z}\left(1+\frac{1}{z^2}\right) \quad \textbf{(9)}$$

Thus, **(6)** holds for n = 0 and 1.

Assume the result is true for $n \le p$.

Then

$$J^2_{p+\frac{1}{2}}(z) + N^2_{p+\frac{1}{2}}(z) = \frac{2}{\pi z}\sum_{k=0}^{p}\frac{(2k-1)!!(p+k)!}{2^k z^{2k} k!(p-k)!}.$$

We have, from **(2)** $\times$**(3)** + **(4)** $\times$**(5),**

$$N^2_{n-\frac{1}{2}}(z) + J^2_{n-\frac{1}{2}}(z) - N^2_{n+\frac{3}{2}}(z) - J^2_{n+\frac{3}{2}}(z) = \frac{2(2n+1)}{z}\left(J_{n+\frac{1}{2}}(z)J'_{n+\frac{1}{2}}(z) + N_{n+\frac{1}{2}}(z)N'_{n+\frac{1}{2}}(z)\right)$$

so that

$$N^2_{n+\frac{3}{2}}(z) + J^2_{n+\frac{3}{2}}(z) = N^2_{n-\frac{1}{2}}(z)\, J^2_{n-\frac{1}{2}}(z) - \frac{2(2n+1)}{z}\left(J_{n+\frac{1}{2}}(z)j'_{n+\frac{1}{2}}(z) + N_{n+\frac{1}{2}}(z)N'_{n+\frac{1}{2}}(z)\right)$$

$$= N^2_{n-\frac{1}{2}}(z) + J^2_{n-\frac{1}{2}}(z) - \frac{2n+1}{z}\,\frac{d}{dz}\left(J^2_{n+\frac{1}{2}}(z) + N^2_{n+\frac{1}{2}}(z)\right).$$

Now, substituting *p* in place of *n*, we have

$$N^2_{p+\frac{3}{2}}(z) + J^2_{p+\frac{3}{2}}(z) = N^2_{p-\frac{1}{2}}(z) + J^2_{p-\frac{1}{2}}(z) - \frac{(2n+1)}{z}\,\frac{d}{dz}\left(J^2_{p+\frac{1}{2}}(z) + N^2_{p+\frac{1}{2}}(z)\right)$$

$$= \frac{2}{\pi z}\sum_{k=0}^{p-1}\frac{(2k-1)!!(p-1+k)!}{2^k z^{2k} k!(p-1-k)!} - \frac{(2p+1)}{z}\,\frac{d}{dz}\left(\frac{2}{\pi z}\sum_{k=0}^{p}\frac{(2k-1)!!(p+k)!}{2^k z^{2k} k!(p-k)!}\right)$$

$$= \frac{2}{\pi z}\left(\sum_{k=0}^{p-1}\frac{(2k-1)!!(p-1+k)!}{2^k z^{2k} k!(p-1-k)!} + \sum_{k=0}^{p}\frac{(2k-1)!!(p+k)!}{2^k z^{2k+2} k!(p-k)!}\,(2p+1) + \sum_{k=1}^{p}\frac{(2k-1)!!(p+k)!(2k)}{2^k z^{2k+2} k!(p-k)!}\,(2p+1)\right)$$

$$= \frac{2}{\pi z}\left(\sum_{k=2}^{p-1}\frac{(2k-1)!!(p-1+k)!}{2^k z^{2k} k!(p-1-k)!} + 1 + \frac{p(p-1)}{2z^2} + (2p+1)\sum_{k=1}^{p}\frac{(2k-1)!!(p+k)!}{2^k z^{2k+2} k!(p-k)!} + \frac{2p+1}{z^2}\right.$$

$$\left. + (2p+1)\sum_{k=1}^{p}\frac{(2k-1)!!(p+k)!(2k)}{2^k z^{2k+2} k!(p-k)!}\right) \qquad \textbf{(10)}$$

$$= \frac{2}{\pi z}\left(1 + \frac{(p+1)(p+2)}{2z^2} + \sum_{k=2}^{p-1}\frac{(2k-1)!!(p-1+k)!}{2^k z^{2k} k!(p+1-k)!} + (2p+1)\sum_{k=2}^{p+1}\frac{(2k-3)!!(p+k-1)!}{2^{k-1} z^{2k} (k-1)!(p+k-1)!}\right.$$

$$\left. + (2p+1)\sum_{k=2}^{P+1}\frac{(2k-3)!!(p+k-1)!(2k-2)}{2^{k-1} z^{2k} (k-1)!(p-k-1)!}\right) \qquad \textbf{(11)}$$

by putting $k+1$ in place of *k* in the second and third summations. Also, these two summations simplify to

$(2p+1)\sum_{k=2}^{p+1}\frac{(2k-1)!!(p+k-1)!}{2^{k-1}z^{2k}(k-1)!(p-k+1)!}$ so that the right hand side of **(11)** is

$$\frac{2}{\pi z}\left(1+\frac{(p+1)(p+2)}{2z^2}+\sum_{k=2}^{p-1}\frac{(2k-1)!!(p-1+k)!}{2^k z^{2k}k!(p-1-k)!}+(2p+1)\sum_{k=2}^{p+1}\frac{(2k-1)!!(p+k-1)!}{2^{k-1}z^{2k}(k-1)!(p-k+1)!}\right)$$

$$=\frac{2}{\pi z}\left(1+\frac{(p+1)(p+2)}{2z^2}+\sum_{k=2}^{p-1}\frac{(2k-1)!!(p-1+k)!}{2^k z^{2k}k!(p+1-k)!}[(p-k+1)(p-k)+2k(2p+1)]\right.$$

$$\left.+(2p+1)\frac{(2p-1)!!(2p-1)!}{2^{p-1}z^{2p}(p-1)!}+(2p+1)\frac{(2p+1)!!(2p)!}{2^p z^{2p+2}p!}\right) \qquad \textbf{(12)}$$

The summation here simplifies to $\sum_{k=2}^{p-1}\frac{(2k-1)!!(p+1+k)!}{2^k z^{2k}k!(p+1-k)!}$ and each of the last two terms simplifies to $\frac{(2p-1)!!(2p+1)!}{2^p z^{2p}p!}$ and $\frac{(2p+1)!!(2p+1)!}{2^p z^{2p+2}p!}$.

Hence, the expression **(11)** is equal to

$$\frac{2}{\pi z}\sum_{k=0}^{p+1}\frac{(2k-1)!!(p+1+k)!}{2^k z^{2k}k!(p+1-k)!}.$$

Thus,

$$N^2_{p+\frac{3}{2}}(z)+J^2_{p+\frac{3}{2}}(z)=\frac{2}{\pi z}\sum_{k=0}^{p+1}\frac{(2k-1)!!(p+1+k)!}{2^k z^{2k}k!(p+1-k)!}$$

so that the result is true for $n = p + 1$.

Hence, by the principle of Mathematical induction, the result is true for all non-negative integral *n*. The formula

$$\sum_{l=0}^{n+\frac{1}{2}}\sum_{k=0}^{n+\frac{1}{2}}\frac{(-1)^l(2k-1)!!(2l-1)!!(n+k)!(n+l)!}{k!l!(n-k)1(n-l)!(2iz)^{l+k}}=\sum_{p=0}^{n}\frac{(2p-1)!!(n+p)!}{2^p z^{2p}p!(n-p)!}$$

follows at once from (1).

RESULTS

We have obtained exact formula for the sum of the squares of the Bessel function and the Neumann function of the same order of half-odd integer. The proof given this article is used to express the above sum as a double summation of two finite series which is a deduction from our proof.

DISCUSSION

The main objective of establishing a formula for the sum of the squares of Bessel and Neumann function of the same order of half-odd integers in to solve the long standing mathematical problem posed by the anomalous absorption of light ion partial wave by the nuclear optical potential. By the exact formula for the sum of the squares of Bessel and Neumann functions of the half-odd integral order, one obtains the exact value of the wave function the asymptotic region. This result can directly be used in case of neutrons scattering on composite nuclei. We believe that the exact formula we established will be very important in mathematics as well.

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